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SOME PROPERTIES OF SPECIFIC INTEGRALS OF THE CAUCHY TYPE AND THEIR APPLICATIONS

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SOME PROPERTIES OF SPECIFIC INTEGRALS OF THE CAUCHY TYPE AND THEIR APPLICATIONS

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ABSTRACT. This paper deals with the study of limited cases of the Plemel'-Privalow theorem. The results are applied to the solution concerning structural properties of the Riemann boundary value problem and the singular integral equation of normal type.

Let us take function f(t), determined on a simple, closed /793* Lyapunov curve L, situated in a complex plane with the origin of coordinates inside L, and examine integral $Sf=\frac{1}{\pi i}\int_L \frac{f(t)\,dt}{t-t_0},\ t_0\in L,$ to be understood as improper in the sense of the principal value according to Cauchy. We shall introduce classes of continuous functions $H(p+\alpha, L)$, p-integral, $p\geqslant 0,\ 0\leqslant\alpha\leqslant 1$. Function $f(t)\in H(p+\alpha, L)$, if the p-derivative f(t) satisfies the condition on L

$$f^{(p)}(t_1)-f^{(p)}(t_2)=O(|t_1-t_2|^{\alpha}),\quad t_1,\ t_2\in L.$$

The Plemel'-Privalov theorem is a well-known one [1], according to which, $Sf\in H(p+\alpha, L)$ results from $f\in H(p+\alpha, L)$, and conversely, if $0<\alpha<1$. When $\alpha=0,\ 1$ this is not true.

The present paper is devoted to the study of limited cases of the Plemel'-Privalov theorem (α = 0, 1). The results obtained are further applied to the problem concerning structural properties of the solution to the Riemann boundary value problem and of the singular integral equation of the normal type.

Let Z(p, L) represent a class of functions determined on L, whose p-derivative satisfies the correlation

$$f^{(p)}[t,\ (s+h)]+f^{(p)}[t\,(s-h)]-2f^{(p)}[t\,(s)]=O(h),$$

s is arc abscissa of the point $t(s)\in L$. We shall designate by $\rho_n(f, L)$ the value of the best approximation of function f, given on L, with the assistance of polynomials of n-degree along the positive and negative degrees t:

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*Numbers in the margin indicate pagination in the foreign text.

$$\rho_n(f, L) = \sup_{P_n \in H_n} \inf_{t \in L} |f(t) - P_n(t)|,$$

where H_n is the aggregate of all polynomials of the type

$$\sum_{-n}^{n} a_k t^k.$$

Theorem 1. If $f \in Z(p, Z)$, then $Sf \in Z(p, L)$, and reciprocally.

Theorem 2. If $f \in Z(p, L)$, then $\rho_n(f, L) = O(n^{-p-1})$, and reciprocally.

Theorem 3. If $\rho_n(f, L) = O(n^{-p-\alpha})$, then $\rho_n(Sf, L) = O(n^{-p-\alpha})$, $0 < \alpha \leqslant 1$, and reciprocally.

Theorem 4. If $f \in C$, i.e., continuous on L, then $\exp(Sf) \in L_p$, $p > 0$, i.e., summed on L at any degree $p > 0$.

Demonstration of all these theorems is based on the representation

$$Sf = \frac{1}{\pi i} \int_{|w|=1} \frac{f[\psi(w)]}{w - w_0} dw + \frac{1}{\pi i} \int_{|w|=1} f[\psi(w)] \left[\frac{\psi'(w)}{\psi(w) - \psi(w_0)} - \frac{1}{w - w_0} \right] dw,$$

in which $t = \psi(w)$ are boundary values of the function, which conformally represents the exterior of the circle $|w| = 1$ on the exterior L, $t_0 = \psi(w_0)$, and also on the corresponding ones (when L ≡ circumference: $|w| = 1$) or the near results of the authors [2-4]. /794

Let us now analyze the Riemann boundary value problem [1, 5,6]: to find a sectionally holomorphic function of the type $\frac{1}{2\pi i} \int_L \frac{\varphi(\tau)\, d\tau}{\tau - z}$, whose angular boundary values φ^+ and φ^- respectively, from within and from outside L, satisfy the condition

$$\varphi^+ = G\varphi^- + f \tag{1}$$

almost anywhere on L, where G is continuous and $f \in L_r$, $r > 1$, on L, index $\varkappa = \frac{1}{2\pi} [\arg G]_L = 0$.

Let us consider two functions identical, if their values coincide almost anywhere on L. Following the arguments of F. D. Gakhov [5], we present G in the form of $G = \psi^+ / \psi^-$, where

$$\psi^{\pm} = \exp[\pm 1/2 \ln G + 1/2 S(\ln G)] \tag{2}$$

and $(\psi^{\pm})^{-1}$ by virtue of theorem 4 enters into L_p with any $p > 0$. After this, we impart to the correlation (1) the appearance

$$\frac{\varphi^+}{\psi^+} - \frac{\varphi^-}{\psi^-} = \frac{f}{\psi^+}. \tag{3}$$

By virtue of Gel'der's inequality $\frac{f}{\psi^+} \in L_{r-\varepsilon}$, where ε is any arbitrarily small positive quantity. By solving problem (3) (see [6]), we obtain

$$\varphi^{\pm} = \psi^{\pm}\left[\pm\frac{1}{2}\frac{f}{\psi^+} + \frac{1}{2}S\left(\frac{f}{\psi^+}\right)\right]. \tag{4}$$

Thus, with the assumptions made there is only one solution to the Riemann problem, contained in $L_{r-\varepsilon}$*.

By modifying and generalizing the arguments mentioned, one can likewise analyze the Riemann problem, when index $0 \neq \varkappa$ and when f(t) is a sectionally continuous function with a limited number of discontinuity points of first kind. One can also use the obtained results for the purpose of analyzing the total, singular, integral equation of the normal type.

It follows from the theorem 3 and formulas (2) and (4): if $\rho_n(G, L)$ and $\rho_n(f, L)$ are of the order of $n^{-p-\alpha}$ (it will be so, if, for example, G and f are contained in $H(p+\alpha, L)$), then $\rho_n(\varphi^{\pm}, L)$ they also are of the order $n^{-p-\alpha}$. This last assertion may be of use in analyzing the rate of convergence of the approximate solution of the singular integral equation [7].

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*I. B. Simonenko has shown that in the case where r = 2 in the formulated deduction ε can be discarded.

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